Expressing Additives Using Multiplicatives and Subexponentials

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Subexponential logic is a variant of linear logic with a family of exponential connectives—called subexponentials—that are indexed and arranged in a pre-order. Each subexponential has or lacks associated structural properties of weakening and contraction. We show that a classical propositional multiplicative subexponential logic (MSEL) with one unrestricted and two linear subexponentials can encode the halting problem for two register Minsky machines, and is hence undecidable. We then show how the additive connectives can be directly simulated by giving an encoding of propositional multiplicative additive linear logic (MALL) in an MSEL with one unrestricted and four linear subexponentials.

1. Introduction

The decision problem for classical propositional multiplicative exponential linear logic (MELL), consisting of formulas constructed from propositional atoms using the connectives \{\otimes, 1, \&\}, is perhaps the longest standing open problem in linear logic. MELL is bounded below by the purely multiplicative fragment (MLL), which is decidable even in the presence of first-order quantification, and above by MELL with additive connectives (MAELL), which is undecidable even for the propositional fragment (Lincoln et al., 1992). This paper tries to make the undecidable upper bound a bit tighter by considering the question of the decision problem for a family of propositional multiplicative subexponential logics (MSEL) (Nigam, 2009; Nigam and Miller, 2009), each of which consists of formulas constructed from propositional atoms using the (potentially infinite) set of connectives \{\otimes, 1, \&\} \cup \bigcup_{u \in \Sigma} \{!, ?^u\}, where \Sigma is a pre-ordered set of subexponential labels, called a subexponential signature, that is a parameter of the family of logics. In particular, we show that a particular MSEL with a subexponential signature consisting of exactly three labels can encode a two register Minsky machine (2RM), which is Turing-equivalent. This is the same strategy used in (Lincoln et al., 1992) to show the undecidability of MAELL, but the encoding in MSEL is different—simpler—for the branching instructions, and shows that additive behavior is not essential to implement testing for zero, which is the operator that appears to be difficult—likely impossible—to encode in ordinary MELL.

This simple demonstration of undecidability raises an obvious proof-theoretic question:
are the additive connectives redundant with multiplicatives and subexponentials? Recall that, with the usual unrestricted exponential connectives of MAELL, certain compound connectives with additive sub-components can be equivalently expressed without the additive connectives: $ !(A \& B) \equiv !(A \otimes B)$ and $ ?(A \oplus B) \equiv ?A \wedge \neg ?B. The standard proofs of these equivalences make use of the contraction and weakening rules for $?$, and are therefore not suitable for the situation where $!$ and $?$ do not enjoy these structural properties. Nevertheless, these equivalences encode an essential insight about how they may be implemented. To illustrate, if we were able to interpret the $!$ connective as a label for the current context, then the implication $ !A \otimes !B \rightarrow !(A \& B)$ can be read as:

To prove $A \& B$ in a given context, separately prove $A$ and $B$ each in a copy of the context.

We will show how the subexponential connectives can be used to build the operation of copying a context. We use a fairly obvious strategy: to copy a context we need to run a computation that duplicates each element of the context until quiescence, that is, until every available formula has been copied. This much can be done with ordinary linear logic. What subexponentials add is the ability to detect when the copying computation is finished, and then, and only then, to progress to the next step. The full MALL proof system is encoded in terms of these staged quiescent computations.

This short paper is organized as follows: in section 2 we sketch the one-sided sequent formulation of MSEL and recall the definition of a 2RM. In section 3 we encode the transition system of a 2RM in a MSEL with a particular signature. In section 4 we argue that the encoding is adequate, i.e., that the halting problem for a 2RM is reduced to the proof search problem for this MSEL-encoding, by appealing to a focused sequent calculus for MSEL. Then, in section 5 we give an encoding of MALL in a different instance of MSEL, and show that it adequately captures MALL proofs. The final section 6 discusses some consequences.

Note: We use the classical dialect of linear logic to show these results. The intuitionistic dialect has the same decision problem because it is possible to faithfully encode (i.e., linearly simulate the sequent proofs of) the classical dialect in the intuitionistic dialect without changing the signature (Chaudhuri, 2010). This paper is an extended version of (Chaudhuri, 2014).

2. Background

2.1. Propositional Subexponential Logic

Let us quickly recall propositional subexponential logic (SEL) and its associated sequent calculus proof system. This logic is sometimes called subexponential linear logic (SELL), but since it is possible for the subexponentials to have linear semantics it is redundant to include both adjectives. Formulas of SEL $(A, B, \ldots)$ are built from atomic formulas $(a, b, \ldots)$ according to the following grammar:

\[
A, B, \ldots ::= a \mid A \otimes B \mid 1 \mid A \oplus B \mid 0 \mid \neg a \mid A \wedge B \mid \perp \mid A \& B \mid \top \mid ?uA \mid !uA
\]
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Each column in the grammar above is a De Morgan dual pair. A positive formula (depicted with \( P \) or \( Q \) when relevant) is a formula belonging to the first line of the grammar, and a negative formula (depicted with \( N \) or \( M \)) is a formula belonging to the second line. The labels \( (u, v, \ldots) \) on the subexponential connectives \( !^n \) and \( ?^n \) belong to a subexponential signature defined below. The additive fragment of this syntax is just used in this section for illustration; we will not be using the additives in our encodings. The fragment without the additives will be called multiplicative subexponential logic \( (MSEL) \).

**Definition 1.** A subexponential signature \( \Sigma \) is a structure \( \langle \Lambda, U, \leq \rangle \) where:

- \( \Lambda \) is a countable set of labels;
- \( U \subseteq \Lambda \), called the unbounded labels; and
- \( \leq \subseteq \Lambda \times \Lambda \) is a pre-order on \( \Lambda \) i.e., it is reflexive and transitive—and \( \leq \)-upwardly closed with respect to \( U \), i.e., for any \( u, v \in \Lambda \), if \( u \in U \) and \( u \leq v \), then \( v \in U \).

We will assume an ambient signature \( \Sigma \) unless we need to disambiguate particular instances of \( MSEL \), in which case we will use \( \Sigma \) in subscripts. For instance, \( MSEL_\Sigma \) is a particular instance of \( MSEL \) for \( \Sigma \).

The true formulas of \( MSEL \) are derived from a sequent calculus proof system consisting of sequents of the form \( \vdash A_1, \ldots, A_n \) (with \( n > 0 \)) and abbreviated as \( \vdash \Gamma \). The contexts \( (\Gamma, \Delta, \ldots) \) are multi-sets of formulas of \( SEL \), and \( \Gamma, \Delta \) and \( \Gamma, A \) stand as usual for the multi-set union of \( \Gamma \) with \( \Delta \) and \( \{A\} \), respectively. The inference rules for \( SEL \) sequents are displayed in figure 1. Most of the rules are shared between \( SEL \) and linear logic and will not be elaborated upon here. The differences are with the subexponentials, for which we use the following definition.

**Definition 2.** For any \( n \in N \) and lists \( \vec{u} = [u_1, \ldots, u_n] \) and \( \vec{A} = [A_1, \ldots, A_n] \), we write \( ?^n \vec{A} \) to stand for the context \( ?^{u_1} A_1, \ldots, ?^{u_n} A_n \). For \( \vec{v} = [v_1, \ldots, v_n] \), we write \( u \leq \vec{v} \) to mean that \( u \leq v_1, \ldots, u \leq v_n \).

The rule for \( ! \), sometimes called promotion, has a side condition that checks that the label of the principal formula is less than the labels of all the other formulas in the context. This rule cannot be used if there are non-?-formulas in the context, nor if the labels of some of the ?-formulas are strictly smaller or incomparable with that of the principal \(!\)-formula. Both these properties will be used in the encoding in the next section. The
structural rules of weakening and contraction apply to those principal \(7\)-formulas with unbounded labels.

2.2. Two Register Minsky Machines

Like Turing machines, Minsky register machines have a finite state diagram and transitions that can perform I/O on some unbounded storage device, in this case a bank of registers that can store arbitrary natural numbers. We shall limit ourselves to machines with two registers (2RM) \(a\) and \(b\), which are sufficient to encode Turing machines.

**Definition 3.** A 2RM is a structure \(\langle Q, *, C, \rightarrow \rangle\) where:

- \(Q\) is a non-empty finite set of states;
- \(* \in Q\) is a distinguished halting state;
- \(C\) is a set of configurations, each of which is a structure of the form \(\langle q, v \rangle\), where \(q \in Q\) and \(v : \{a, b\} \rightarrow N\), that assigns values (natural numbers) to the registers \(a\) and \(b\) in state \(q\);
- \(\rightarrow \subseteq C \times I \times C\) is a deterministic labeled transition relation between configurations where the label set \(I = \{\text{halt}, \text{incra}, \text{incrb}, \text{decrb}, \text{isza}, \text{iszrb}\}\) (called the instructions).

By usual convention, we write \(\rightarrow\) infix with the instruction atop the arrow. We require that every element of \(\rightarrow\) fits one of the following schemas, where in each case \(q, r \in Q\) and \(q \neq r\):

\[
\begin{align*}
\langle q, v \rangle \xrightarrow{\text{halt}} &\langle *, \{a:0, b:0\}\rangle & (\text{with } q \neq *) \\
\langle q, \{a:m, b:n\}\rangle \xrightarrow{\text{incra}} &\langle r, \{a:m+1, b:n\}\rangle \\
\langle q, \{a:m, b:n\}\rangle \xrightarrow{\text{incrb}} &\langle r, \{a:m, b:n+1\}\rangle \\
\langle q, \{a:m+1, b:n\}\rangle \xrightarrow{\text{decrb}} &\langle r, \{a:m, b:n\}\rangle \\
\langle q, \{a:m, b:n+1\}\rangle \xrightarrow{\text{decrb}} &\langle r, \{a:m, b:n\}\rangle \\
\langle q, \{a:0, b:n\}\rangle \xrightarrow{\text{isza}} &\langle r, \{a:0, b:n\}\rangle \\
\langle q, \{a:m, b:0\}\rangle \xrightarrow{\text{iszrb}} &\langle r, \{a:m, b:0\}\rangle \\
\end{align*}
\]

For a trace \(\vec{i} = [i_1, \ldots, i_n]\), we write \(\langle q_0, v_0 \rangle \xrightarrow{\vec{i}} \langle q_n, v_n \rangle\) if \(\langle q_0, v_0 \rangle \xrightarrow{i_1} \cdots \xrightarrow{i_n} \langle q_n, v_n \rangle\).

The 2RM halts from an initial configuration \(\langle q_0, v_0 \rangle\) if there is a trace \(\vec{i}\) such that \(\langle q_0, v_0 \rangle \xrightarrow{\vec{i}} \langle *, \{a:0, b:0\}\rangle\). (The configuration \(\langle *, \{a:0, b:0\}\rangle\) will be called the halting configuration.) The halting problem for a 2RM is the decision problem of whether the machine halts from an initial configuration.

The requirement that \(\rightarrow\) be deterministic amounts to: \(\langle q, v \rangle \xrightarrow{i} \langle q_1, v_1 \rangle\) and \(\langle q, v \rangle \xrightarrow{j} \langle q_2, v_2 \rangle\) imply that \(i = j\), \(q_1 = q_2\), and \(v_1 = v_2\). Note that a trace that does not end with a halting configuration will not be considered to be halting, even if there is no possible successor configuration. It is an easy exercise to transform a given 2RM into one where every configuration has a successor except for the halting configuration.
Theorem 1 ((Minsky, 1961)). The halting problem for 2RM is recursively unsolvable.

\[ \square \]

3. Encoding a 2RM

For a given 2RM, which we fix in this section, we will encode its halting problem as the derivability of a particular MSEL sequent that encodes its labeled transition system and the initial configuration. We will use the following subexponential signature in the rest of this section.

**Definition 4.** Let \( \Xi \) stand for the signature \( \langle \{ \infty, a, b \}, \{ \infty \}, \leq \rangle \) where \( \leq \) is the reflexive-transitive closure of \( \leq_0 \) defined by \( a \leq_0 \infty \) and \( b \leq_0 \infty \).

**Definition 5 (encoding configurations).** For \( c = \langle q, v \rangle \), we write \( \mathcal{E}(c) \) for the following MSEL\(\Xi\) context:

\[
\begin{array}{c}
\neg b \neg ra, \neg a ra, \neg b rb, \neg b rb, \ldots, \neg a ra, \neg b rb, \neg q \\
\text{length} = \nu(a) \\
\text{length} = \nu(b)
\end{array}
\]

**Definition 6 (encoding transitions).** The transitions (1) of the 2RM are encoded as a context \( \Pi \) with:

- to represent \( \langle q, v \rangle \xrightarrow{\text{halt}} \langle s, \{ a:0, b:0 \} \rangle \), the elements: \( q \otimes \neg h, h \otimes !b rb \otimes \neg h, h \otimes \not\in \Xi \{ \infty \} \) (for some \( h \not\in Q \));
- to represent \( \langle q, \{ a: m, b: n \} \rangle \xrightarrow{\text{incra}} \langle r, \{ a: m+1, b: n \} \rangle \), the element \( q \otimes (\neg r \not\Xi \neg \infty a ra) \);
- to represent \( \langle q, \{ a: m, b: n \} \rangle \xrightarrow{\text{incrb}} \langle r, \{ a: m, b: n+1 \} \rangle \), the element: \( q \otimes (\neg r \not\Xi \neg \infty b rb) \);
- to represent \( \langle q, \{ a: m+1, b: n \} \rangle \xrightarrow{\text{decra}} \langle r, \{ a: m, b: n \} \rangle \), the element: \( q \otimes !b ra \otimes \neg r \);
- to represent \( \langle q, \{ a: m, b: n+1 \} \rangle \xrightarrow{\text{decrb}} \langle r, \{ a: m, b: n \} \rangle \), the element: \( q \otimes !b rb \otimes \neg r \);
- to represent \( \langle q, \{ a: 0, b: n \} \rangle \xrightarrow{\text{inzb}} \langle r, \{ a: 0, b: n \} \rangle \), the element: \( q \otimes !b ra \otimes \neg r \); and
- to represent \( \langle q, \{ a: m, b: 0 \} \rangle \xrightarrow{\text{inzb}} \langle r, \{ a: m, b: 0 \} \rangle \), the element: \( q \otimes !a \neg r \).

Note that \( \Pi \) contains a finite number of elements.

**Definition 7 (encoding the halting problem).** If \( \Gamma \) is \( A_1, \ldots, A_n \), then let \( ?^n \Gamma \) stand for \( ?^n A_1, \ldots, ?^n A_n \). The encoding of the halting problem for the 2RM from the initial configuration \( c_0 = \langle q_0, v_0 \rangle \) is the MSEL\(\Xi\) sequent \( \vdash ?^\infty \Pi, \mathcal{E}(c_0) \).

**Theorem 2.** If the 2RM halts from \( c_0 \), then \( \vdash ?^\infty \Pi, \mathcal{E}(c_0) \) is derivable.

**Proof.** We will show that if \( c = \langle q_1, v_1 \rangle \xrightarrow{i} \langle q_2, v_2 \rangle = d \) (for some \( i \)), then the following MSEL\(\Xi\) rule is derivable:

\[
\begin{align*}
\vdash ?^\infty \Pi, \mathcal{E}(d) \\
\vdash ?^\infty \Pi, \mathcal{E}(c)
\end{align*}
\]

This is largely immediate by inspection. Here are three representative cases.
The case of $i = \text{incra}$: it must be that $v_2(a) = v_1(a) + 1$ and $v_2(b) = v_1(b)$, so $E(d) = E(c) \setminus \{\neg q_1\}, \neg q_2, \neg^a ra$. Moreover, $q_1 \otimes \neg q_2 \gamma \neg^a ra \in \Pi$. So:

$$
\vdash \neg^\Pi, E(c) \setminus \{\neg q_1\}, \neg q_2, \neg^a ra
$$

The cases for $\text{incrb}$, $\text{decrb}$, and $\text{decrb}$ are similar.

The instance of $!$ is justified because $b \leq \infty$ and $b \leq h$, and there are no $?$-formulas labeled $a$ or non-$?$ formulas in the sequent. The case of $\text{iszb}$ is similar.

The case of $i = \text{halt}$. Here, we know that $q_1 \otimes h \in \Pi$, so:

$$
\vdash \neg^\Pi, E(c) \setminus \{\neg q_1\}, \neg h
$$

Now, as long as there are any occurrences of $\neg^a ra$ or $\neg^a rb$ in $E(c)$, we can apply one of the decrementing rules $h \otimes !^b ra \otimes \neg h$ or $h \otimes !^b rb \otimes \neg h \in \Pi$. The general case looks something like this, where $\Delta_{ra} = \{\neg ra, \ldots, \neg ra\}$ and $\Delta_{rb} = \{\neg rb, \ldots, \neg rb\}$.

There is a symmetric case for contracting the $h \otimes !^b rb \otimes \neg h$. Eventually, the right branch just becomes $\vdash \neg^\Pi, \neg h$, at which point we have:

$$
\vdash \neg^\Pi, \neg h
$$
4. Adequacy of the Encoding via Focusing

By the contrapositive of theorem 2, if the sequent $\vdash_{\Xi} \gamma^\pi \Pi, \mathcal{E}(c_0)$ is not derivable, then the 2RM does not halt from $c_0$. This gives half of the reduction. For the converse of theorem 2, we need to show how to recover a halting trace by searching for proofs of a MSEL encoding of a halting problem. The best way to do this is to build a focused proof which will have the derived inference rules in the above proof as the only possible synthetic rules, in a sense made precise below. We will begin by sketching the focused proof system for SEL that is sound and complete for the unfocused system of figure 1, and then show how the synthetic rules for the encoding are in bijection for all instructions (with a small correction needed for \textit{halt}).

Focusing is a general technique to restrict the non-determinism in a cut-free sequent proof system. Though originally defined for classical linear logic in (Andreoli, 1992), it is readily extended to many other logics (Chaudhuri et al., 2008; Liang and Miller, 2009; Nigam, 2009). This section sketches the basic design of a focused version of the rules of figure 1, and omits most of the meta-theoretic proofs of soundness and completeness, for which the general proof techniques are by now well known (Chaudhuri et al., 2008; Miller and Saurin, 2007; Simmons, 2014). To keep things simple, we will define a focused calculus by adding to the unfocused system a new kind of \textit{focused sequent}, $\vdash_{\Omega} [A]$, where the formula $A$ is \textit{under focus}. Contexts written with $\Omega$, which we call \textit{neutral contexts}, can contain only positive formulas, atoms, negated atoms, and $?$-formulas. The rules of the focused proof system for SEL are depicted in figure 2.

Focused sequents are created—reading from conclusion upwards to premises—from unfocused sequents with neutral contexts by means of the rules \textit{decide}, \textit{idecide}, or \textit{udecide}. In a focused sequent, only the formula under focus can be principal, and the focus persists on the immediate subformulas of this formula in the premises, with the exception of the rule \textit{!}. In the base case, for \textit{[init]}, the focused atom must find its negation in the context, while all formulas in the context must be $?$-formulas with unbounded labels. When the focused formula is negative, the focus is released with the \textit{[blur]} rule, at which point any of the unfocused rules \{$\forall$, $\perp$, $\&$, $\top$\} of figure 1 can be used to decompose the formula and its descendants further. Eventually, when there are no more negative descendants—\textit{i.e.}, the whole context has the form $\Omega$—a new focused phase is launched again and the cycle
repeats. Note that the structural rules contr and weak of the unfocused calculus are removed in the focused system. Instead, weakening is folded into [init], [!] and [1], and contraction is folded into [⊗] and udecide. The rules contr and weak remain admissible for either sequent form in the focused calculus.

**Theorem 3.** The SEL sequent \( \vdash \Gamma \) is provable in the unfocused system of figure 1 if and only if it is provable in the focused system of figure 2.

**Proof.** Straightforward adaptation of existing proofs of the soundness and completeness of focusing, such as (Chaudhuri et al., 2008; Miller and Saurin, 2007; Simmons, 2014). An instance for SEL can be found in (Nigam, 2009, chapter 5).

**Theorem 4.** The 2RM halts from \( c_0 \) if \( \vdash \Xi ?^\infty \Pi, \mathcal{E}(c_0) \) is derivable.

**Proof.** We will show instead that the 2RM halts from \( c_0 \) if the sequent \( \vdash \Xi ?^\infty \Pi, \mathcal{E}(c_0) \) is derivable in the focused calculus, and we will moreover extract the halting trace from such a focused proof. The required result will then follow immediately from theorem 3, since any provable SEL sequent has a focused proof.

Let a focused proof of \( \vdash \Xi ?^\infty \Pi, \mathcal{E}(c) \) (for \( c = \langle q, v \rangle \)) be given. We proceed by induction on the lowermost instance of udecide in this proof. Note that the MSEL\( \Xi \) context \( ?^\infty \Pi, \mathcal{E}(c) \) is neutral; moreover, all the elements of \( \mathcal{E}(c) \) are either negated atoms or \( ? \)-prefixed negated atoms with bounded labels. So, the only rules of the focusing system that apply to this sequent are ldecide or udecide. However, if we use ldecide, then the premise becomes unprovable, as there is no way to remove an occurrence of \( \neg r_a \) or \( \neg r_b \) from a context that also contains \( \neg q \). Thus, the only possible rule will be an instance of udecide, with the focused formula in the premise being one of the \( \Pi \). First, consider the case where the focused formula does not contain \( h \), i.e., it corresponds to one of the instructions in \( I \setminus \{ \text{halt} \} \). In each of these cases, the focused phase that immediately follows is deterministic. As a characteristic case, suppose the focused formula is \( q \otimes \neg r \); then we have:

\[
\begin{align*}
\vdash \neg q, [q] & \quad [\text{init}] \quad \vdash \Xi ?^\infty \Pi, \mathcal{E}(c) \setminus \{ \neg q \}, \neg r \quad [\otimes] \\
\vdash ?^\infty \Pi, \mathcal{E}(c) \setminus \{ \neg q \}, [\neg r] & \quad [\text{init}] \\
\vdash ?^\infty \Pi, \mathcal{E}(c) & \quad [\neg r] \quad [\otimes] \\
\vdash \Xi ?^\infty \Pi, \mathcal{E}(c) & \quad \text{udecide}
\end{align*}
\]

The right premise is now itself neutral and an encoding of a different configuration. We can appeal to the inductive hypothesis to find a halting trace for it, to which we can prepend the instruction isza to get the halting trace from \( c \). A similar argument can be used for the other instructions in \( I \setminus \{ \text{halt} \} \).

This leaves just the formulas involving \( h \) for the lowermost udecide. We cannot select any formula but \( q \otimes \neg h \) from \( \Pi \), for the derivation would immediately fail because \( h \not\in Q \) and there is no \( \neg h \) in \( \mathcal{E}(c) \) to use with [init]. So, as the formula selected is \( q \otimes \neg h \), we
have:

\[
\begin{align*}
\vdash \neg q, [q] & \quad \text{[init]} \\
\vdash \neg q, [\neg q], [\neg h] & \quad \text{[blur]} \\
\vdash \neg q, [q \otimes \neg h] & \quad \text{[udecide]} \\
\vdash \neg q & \quad \text{[init]} \\
\vdash \neg q, [\neg q], [\neg h] & \quad \text{[blur]} \\
\vdash \neg q, [q \otimes \neg h] & \quad \text{[udecide]}
\end{align*}
\]

The context of the right premise is now neutral, so the only rule that applies to it is \textit{udecide}. A simple nested induction will show that sequents of this form \( \vdash \Pi, E(c) \setminus \{\neg q\}, \neg h \) are always derivable in the focused calculus. Therefore, the trace that corresponds to the configuration \( c \) is just the singleton \textit{halt}.

\textbf{Corollary 1.} The derivability of \textit{MSEL} sequents is recursively unsolvable.

\textit{Proof.} Directly from theorems 1, 2, and 4. \hfill \square

\section{5. Directly Encoding MALL}

Since \textit{MSEL} is Turing-equivalent, it can obviously be used to simulate a theorem prover that implements a complete search procedure for \textit{MAELL}. Thus, in an indirect fashion, we see that additive behavior can be encoded using multiplicatives and subexponentials alone. In this section we will give a more direct demonstration of this ability by showing how to simulate propositional \textit{MALL} in a propositional \textit{MSEL}.

We use the propositional version to keep the result as strong as possible. As a price, the encoding needs to be specialized to every subformula of the endsequent, and is therefore exponentially bigger than the \textit{MALL} sequent we start with. If we were to use a first-order variant of \textit{MSEL} with the same signature, then our encoding would be polynomial in the size of the \textit{MALL} sequent, since the \textit{theory} of \textit{MALL} would be of constant size. This is indeed the technique used by many of the encodings of various proof systems in \textit{SEL} (Nigam et al., 2014).

\textbf{Definition 8.} Let \( \mathcal{T} \) stand for the signature \( \langle \{\infty, 1, r, lr, lin\}, \{\infty\}, \leq \rangle \) where \( \leq \) is the reflexive-transitive closure of \( \leq_0 \) defined by \( l \leq_0 \infty, r \leq_0 \infty, lr \leq_0 1, lr \leq_0 r \) and \( lin \leq_0 \infty \). In other words, \( \leq \) has the following lattice.

\[
\begin{array}{c}
\infty \\
1 \\
r \\
lr \\
lin
\end{array}
\]

Recall that \textit{MALL} formulas are built from \textit{literals} (atoms or negated atoms) and the connectives \( \{\otimes, 1, \gamma, \bot, \oplus, 0, \&, \top\} \). As this is a sub-language of \textit{SEL}, we use the same inference system as in Figure 1. We will build an \textit{MSEL} sequent whose \textit{MSEL} proofs are in bijection to the \textit{MALL} proofs of \( \vdash \Gamma \).

\textbf{Notation 1.} We write \( A \vDash B \) to indicate that \( A \) is a subformula of \( B \). Likewise \( A \vDash \Gamma \) means that \( A \vDash B \) for some \( B \in \Gamma \).
Definition 9 (atomic formulas). If $\Gamma$ is a multiset of MALL formulas, then we add the following atomic formulas to the collection of atoms used in the MSEL$_\gamma$ encoding of $\Gamma$.

— The atom rule to represent the conclusions and premises of every MALL inference rule.
— For every $C \in \Gamma$, the atom $f_C$ to represent an occurrence of $C$. If $\Gamma$ is the multiset $\{A_1, \ldots, A_n\}$, then we write $f_\Gamma$ to stand for the multiset $\{f_{A_1}, \ldots, f_{A_n}\}$.
— For every $A \oplus B \in \Gamma$, the atom $\text{ch}_{A,B}$ to represent the choice between $A$ and $B$.
— The atoms $\text{cp}$, $\text{restl}$ and $\text{restr}$ to represent stages in the copying procedure for contexts.
— The atom $\text{gc}$ to represent stages in the deletion procedure for contexts.

We assume that none of these atomic formulas occurs as a subformula of $\Gamma$.

We will use the above subexponential signature and collection of atomic formulas to encode the MALL inference system specialized to the endsequent $\vdash \Gamma$. This encoding will be in the form of a theory $\Theta_\Gamma$ that will be a collection of formulas with $!$s and $?$s interspersed to place and check occurrences.

Definition 10 (MALL encoding theory). If $\Gamma$ is a multiset of MALL formulas, then the theory $\Theta_\Gamma$ is made up of the following elements.

— For every atom $a \in \Gamma$,

$$\text{rule} \otimes !^{\text{lin}} \neg f_a \otimes !^{\text{lin}} \neg f_{\neg a}.$$ 

— For every $A \otimes B \in \Gamma$, the formula:

$$\text{rule} \otimes !^{\text{lin}} \neg f_{A \otimes B} \otimes \left( ?^{\text{lin}} f_A \gamma \neg \text{rule} \right) \otimes \left( ?^{\text{lin}} f_B \gamma \neg \text{rule} \right).$$

— If $1 \in \Gamma$, then:

$$\text{rule} \otimes !^{\text{lin}} \neg f_1.$$ 

— For every $A \mp B \in \Gamma$, the formula:

$$\text{rule} \otimes !^{\text{lin}} \neg f_{A \mp B} \otimes \left( ?^{\text{lin}} f_A \gamma \neg \text{rule} \right) \otimes \left( ?^{\text{lin}} f_B \gamma \neg \text{rule} \right).$$

— If $\bot \in \Gamma$, then:

$$\text{rule} \otimes !^{\text{lin}} \neg f_{\bot} \otimes \neg \text{rule}.$$ 

— For every $A \otimes B \in \Gamma$, the formulas:

$$\text{rule} \otimes !^{\text{lin}} \neg f_{A \otimes B} \otimes \neg \text{ch}_{A,B},$$

$$\text{ch}_{A,B} \otimes \left( ?^{\text{lin}} f_A \gamma \neg \text{rule} \right),$$

$$\text{ch}_{A,B} \otimes \left( ?^{\text{lin}} f_B \gamma \neg \text{rule} \right).$$

— For every $A & B \in \Gamma$, the formulas:

$$\text{rule} \otimes !^{\text{lin}} \neg f_{A & B} \otimes \neg \text{cp}.$$
Expressing Additives Using Multiplicatives and Subexponentials

\[ \text{restl} \otimes \text{lin} \left( \text{rule} \right) \]

\[ \text{restr} \otimes \text{lin} \left( \text{rule} \right) \]

To implement copying, the following formula for every \( C \in \Gamma \):

\[ \text{cp} \otimes \text{lin} \left( \text{rule} \right) \]

To detect when copying is finished, the additional formula:

\[ \text{cp} \otimes \text{lr} \left( \text{rule} \right) \]

Once copying is finished, to implement the transferal of formulas from the temporary subexponential labels \( l \) or \( r \) to the main label \( \text{lin} \), the following formulas for every \( C \in \Gamma \):

\[ \text{restl} \otimes \text{lin} \left( \text{rule} \right) \]

\[ \text{restr} \otimes \text{lin} \left( \text{rule} \right) \]

Note that the transferal process ends with one of the latter two steps in the theory elements for \( A \) & \( B \) above.

If \( \top \in \Gamma \), then the following two formulas to initiate and terminate deletion:

\[ \text{rule} \otimes \text{lin} \left( \text{rule} \right) \]

To implement deleting the context during a proof of \( \top \), the following formula for every \( C \in \Gamma \):

\[ \text{gc} \otimes \text{lin} \left( \text{rule} \right) \]

Note: since \( \Gamma \) has finitely many subformulas, it must be that \( \Theta_{\Gamma} \) is finite.

This brings us to the encoding of MALL contexts.

**Definition 11 (encoding MALL contexts).** If \( \Gamma \) is a multiset of MALL formulas, then \( \mathcal{E}(\Gamma) \) stands for the multiset

\[ ?^{\infty}_{\Theta_{\Gamma}} \left\{ ?^{\text{lin}}_{f_C} : C \in \Gamma \right\} \]

We will now prove the following adequacy theorem.

**Theorem 5.** If \( \Gamma_0 \) is a multiset of MALL formulas, then \( \vdash_{\text{MALL}} \Gamma_0 \) if and only if \( \vdash_{\text{MSEL}} \mathcal{E}(\Gamma_0) \).

**Proof.** Let us first show that the encoding can simulate MALL proofs, i.e., if \( \vdash_{\text{MALL}} \Gamma_0 \) then \( \vdash_{\text{MSEL}} \mathcal{E}(\Gamma_0) \). We will establish this by showing that all the inference rules of MALL are simulated by the encoding, i.e., for every MALL inference rule of the form

\[ \vdash \Gamma_1 \quad \ldots \quad \vdash \Gamma_n \]

\[ \vdash \Gamma_0 \]
there is a derived $\text{MSEL}_\Gamma$ inference rule for

\[
\vdash \mathcal{E}(\Gamma_1) \quad \cdots \quad \vdash \mathcal{E}(\Gamma_n) \\
\vdash \mathcal{E}(\Gamma_0)
\]

Note that this proof only makes sense if the $\text{MALL}$ inference system has the subformula property—otherwise, $\mathcal{E}(\Gamma_1), \ldots, \mathcal{E}(\Gamma_n)$ would not be fragments of $\mathcal{E}(\Gamma_0)$—which is the case for us since our proof system (Figure 1) is cut-free. There are the following cases.

— Case of $\text{init}$: here, we know that

\[
\mathcal{E}(a, \neg a) = \gamma^\infty \left( \text{rule} \otimes \ldots \right), \\
\gamma_{\text{lin}} f_a, \gamma_{\text{lin}} f_{\neg a}, \neg \text{rule}.
\]

so:

\[
\text{rule, } \neg \text{rule} \quad \text{init} \\
\vdash \text{rule} \otimes \ldots \quad \text{init} \\
\vdash \gamma^\infty \left( \text{rule} \otimes \ldots \right), \\
\gamma_{\text{lin}} f_a, \gamma_{\text{lin}} f_{\neg a}, \neg \text{rule}
\]

— Case of $\otimes$: we know that:

\[
\mathcal{E}(\Gamma, \Delta, A \otimes B) = \gamma^\infty \Theta_{\Gamma, \Delta, A \otimes B}, \gamma_{\text{lin}} f_{\Theta} \Delta, \gamma_{\text{lin}} f_{\Theta} A, \gamma_{\text{lin}} f_{\Theta} B, \neg \text{rule}.
\]

where

\[
\Theta_{\Gamma, \Delta, A \otimes B} \ni \left( \text{rule} \otimes \ldots \right), \quad \Theta_{\Gamma, \Delta, A \otimes B} \ni \left( \gamma_{\text{lin}} f_{\Theta} \Delta \neg \text{rule} \right), \quad \Theta_{\Gamma, \Delta, A \otimes B} \ni \left( \gamma_{\text{lin}} f_{\Theta} A \neg \text{rule} \right). \quad (R)
\]

Moreover, $\Theta_{\Gamma, \Delta, A \otimes B}$ is the same set of formulas as $\Theta_{\Gamma, \Delta, A \otimes B} \ni \gamma_{\text{lin}} f_{\Theta} \Delta, \gamma_{\text{lin}} f_{\Theta} A, \neg \text{rule}$, so the latter is related to the former by a sequence of contractions. So, we have:

\[
\vdash \gamma^\infty \Theta_{\Gamma, \Delta, A \otimes B}, \gamma_{\text{lin}} f_{\Theta} \Delta, \gamma_{\text{lin}} f_{\Theta} A, \neg \text{rule} \\
\vdash \gamma^\infty \Theta_{\Gamma, \Delta, A \otimes B}, \gamma_{\text{lin}} f_{\Theta} \Delta, \gamma_{\text{lin}} f_{\Theta} A, \neg \text{rule} \\
\vdash \gamma^\infty \Theta_{\Gamma, \Delta, A \otimes B}, \gamma_{\text{lin}} f_{\Theta} \Delta, \gamma_{\text{lin}} f_{\Theta} A, \neg \text{rule}
\]

where the two additional closed branches are not shown.

— The case of $1$ is a trivial analogue of the previous case.

— The cases of $\emptyset$ and $\bot$ are completely straightforward.

— The case of $\otimes$: we know that

\[
\mathcal{E}(\Gamma, A \otimes B) = \gamma^\infty \Theta_{\Gamma, A \otimes B}, \gamma_{\text{lin}} f_{\Theta} \Delta, \gamma_{\text{lin}} f_{\Theta} A, \neg \text{rule}.
\]

where

\[
\Theta_{\Gamma, A \otimes B} \ni \text{rule} \otimes \ldots, \\
\text{ch}_{\Theta} A, B \ni \left( \gamma_{\text{lin}} f_{\Theta} \Delta \neg \text{rule} \right), \quad (C_A) \\
\text{ch}_{\Theta} A, B \ni \left( \gamma_{\text{lin}} f_{\Theta} A \neg \text{rule} \right), \quad (C_B)
\]

Note that both $\Theta_{\Gamma, A}$ and $\Theta_{\Gamma, B}$ are reachable from $\Theta_{\Gamma, A \otimes B}$ by a sequence of weakenings. So, we have the following derivation as one possibility, where many irrelevant details
and closed branches are omitted for lack of space.

\[ \begin{align*}
\vdash ?^\infty \Theta_\Gamma, A, ?^{11n} f_A, ?^{11n} f_A, \neg \text{rule} \\
\vdash ?^\infty \Theta_\Gamma, A, ?^{11n} C_A, ?^{11n} f, \neg \text{ch}_{A,B} \\
\vdash ?^\infty \Theta_\Gamma, A, ?^{11n} C, ?^{11n} f_A, ?^{11n} f_{A&B}, \neg \text{rule}. \\
\end{align*} \]

\[ \text{contr}^*, \text{weak}^*, ? \]

The other possibility is that \( C_A \) is weakened and \( C_B \) is contracted, which is symmetric.

— Case of \&: we know that:

\[ E(\Gamma, A & B) = ?^\infty \Theta_{\Gamma, A&B}, ?^{11n} f_A, ?^{11n} f_{A&B}, \neg \text{rule} \]

where:

\[ \Theta_{\Gamma, A&B} \supset \text{rule} \otimes !^{11n} (\neg f_{A&B} \otimes \neg \text{cp}) \]  
\[ \{ \text{cp} \otimes !^{11n} \neg f_C \otimes \left( ?^* f_C \text{ guard} \neg \text{cp} \right) : C \not\in \Gamma_0 \} \]
\[ \text{cp} \otimes !^{11n} (l^1 \neg \text{restl} \otimes l^r \neg \text{restr}) \]
\[ \{ \text{restl} \otimes !^{11n} \neg f_C \otimes \left( ?^{11n} f_C \text{ guard} \neg \text{restl} \right) : C \not\in \Gamma_0 \} \]
\[ \{ \text{restr} \otimes !^{11n} \neg f_C \otimes \left( ?^{11n} f_C \text{ guard} \neg \text{restr} \right) : C \not\in \Gamma_0 \} \]
\[ \text{restl} \otimes !^{11n} (\neg f_A \text{ guard}) \]
\[ \text{restr} \otimes !^{11n} (\neg f_B \text{ guard}) \]

Moreover, both \( \Theta_{\Gamma, A} \) and \( \Theta_{\Gamma, B} \) are related to \( \Theta_{\Gamma, A&B} \) by contraction and weakening. The derivation begins as follows, where we now also elide the rule names for lack of space, but note just which of \( R_1, \ldots, R_7 \) above were principal.

\[ \begin{align*}
\vdash ?^\infty \Theta_{\Gamma, A}, ?^{11n} f_B, ?^{11n} f_A, \neg \text{rule} & \quad R_6 \\
\vdash ?^\infty \Theta_{\Gamma, B}, ?^{11n} f_A, ?^{11n} f_B, \neg \text{rule} & \quad R_7 \\
\vdash ?^\infty \Theta_{\Gamma, B}, ?^{11n} f_A, ?^{11n} f_B, \neg \text{restl} & \quad R_4 \\
\vdash ?^\infty \Theta_{\Gamma, B}, ?^{11n} f_A, ?^{11n} f_B, \neg \text{restr} & \quad R_5 \\
\vdash ?^\infty \Theta_{\Gamma, B}, ?^{11n} f_A, ?^{11n} f_B, \neg \text{cp} & \quad R_3 \\
\vdash ?^\infty \Theta_{\Gamma, A}, ?^{11n} \Psi, ?^{11n} \Psi, ?^{11n} f, \neg \text{restl} & \quad R_2 \\
\vdash ?^\infty \Theta_{\Gamma, A}, ?^{11n} \Psi, ?^{11n} \Psi, ?^{11n} f, \neg \text{restr} & \quad R_1 \\
\vdash ?^\infty \Theta_{\Gamma, A}, ?^{11n} \Psi, ?^{11n} \Psi, ?^{11n} f, \neg \text{cp} & \quad R_1 \\
\end{align*} \]

There are three crucial points. The first is the inference corresponding to \( R_3 \): this rule requires all the subexponential labels in the context to be above \( 1r \), which is the case since both \( 1r \not\leq 1r \) and \( 1r \leq r \). This rule could not be applied any earlier (lower) because \( 1\text{lin} \not\leq 1r \); it had to wait until the \( R_{2*} \) rules would transfer all the \( ?^{11n} \) formulas. Once this outer test succeeds, we apply the \( \otimes \) rule, and have two inner tests; the left test checks for the emptiness of \( r \) since \( 1 \not\leq r \), and the right test does the converse. This guarantees that all the \( ?^1 \) formulas are indeed sent to the left branch, and the \( ?^2 \) formulas to the right branch. The other two crucial points are the applications corresponding to \( R_6 \) and \( R_7 \). In each case, the \( !^{11n} \) guard checks that the
1 or r formulas (as applicable) are absent since lin $\not\subseteq l$ and lin $\not\subseteq r$. Neither of these rules could have been applied any earlier, while there were still $\gamma^l$ or $\gamma^r$ formulas left to transfer back to lin.

Finally, the case of $\top$: we know that:

$$
\mathcal{E}(\Gamma, \top) = \gamma^{\infty}\Theta_{\Gamma, \top}, \gamma^{\text{lin}}f_{\top}, \gamma^{\text{lin}}f_{\top}, \neg \text{rule}.
$$

where:

$$
\Theta_{\Gamma, \top} \supseteq \text{rule} \otimes !^{\text{lin}}\neg f_{\top} \otimes \neg gc,
$$

$$(R_1)$$

$$\{ gc \otimes !^{\text{lin}}\neg f_{C} \otimes \neg gc : C \varepsilon \Gamma_0 \}.$$  

$$(R_{3, \ast})$$

gc,  

$$(R_2)$$

We have:

$$
\frac{\vdash \neg gc \text{ init}}{\vdash \neg ? R_{3}, \neg gc \text{ R}_{3, \ast}}$$

$$
\frac{\vdash \gamma^{\infty}\{ R_{2}, R_{3, \ast} \}, \gamma^{\text{lin}}f_{\Gamma}, \neg gc \text{ R}_{1}}{\vdash \gamma^{\infty}\Theta_{\Gamma, \top}, \gamma^{\text{lin}}f_{\top}, \gamma^{\text{lin}}f_{\top}, \neg \text{rule} \text{ weak}^*}
$$

As in the previous case, the key rule $R_2$ could not be applied any earlier (lower) if there were any $?^{\text{lin}}$ formulas left, since lin is a linear subexponential.

In the reverse direction, we follow the strategy of Theorem 4 and show that the above derived inferences are the only possible ones in a focused proof system. The argument is fairly straightforward: in each case, the use of the atoms rule, ch, cp, restl, restr, and gc guarantee that only the relevant formulas from $\Theta_{\Gamma_0}$ can be decided on; any other decision would immediately fail. The f atoms further limit the selection of the elements of $\Theta_{\Gamma_0}$ to those that are the encoding of the correct principal formulas. As already argued above, the computations in $\oplus$, $\&$, and $\top$ are carefully guarded with $!$s to prevent any interleaving and to ensure the correct distribution of the linear resources; any other distribution would lead to a failure within the same phase of focusing. Thus, from a focused MSEL proof of $\vdash \mathcal{E}(\Gamma_0)$, if we abstract the proof to keep only those sequents that contain $\neg \text{rule}$, and then erase everything except the f-atoms, we would get a proof tree that is isomorphic to a MALL proof of $\vdash \Gamma_0$.

6. Perspectives

The conclusion of Section 4 is that it is not the additives that make MAELL undecidable but rather the fact that the additives can be used to implement a kind of limited test for a portion of the linear resources. Thus, if we have access to such tests using other logical means, then the decision problem can still be undecidable. Section 5 further shows that if we have such alternative tests, then we can in fact implement the additive connectives. Thus, these alternative tests are in fact stronger than the additive connectives. A natural question then is: are the mechanisms provided by subexponentials strictly stronger than...
additives? Once again, there is an indirect negative answer in the sense of implementing a complete theorem prover for a MSEL using the Turing-equivalence of MAELL. It would be good to have a more direct encoding of any MSEL in MAELL, but it is not obvious how to construct such encodings.

We submit the two technical theorems in Sections 4 and 5 as arguments for a philosophical view of MSEL as a logic of ordinary computers. A standard computer—with a Von Neumann architecture, say, but this is not important—is manifestly able to maintain a bank of counters and to observe and act upon the value of any particular counter. It is unnatural, though, to allow such a computer to take computational steps that depend on a different computation in a parallel reality that is identical to the present reality in all but a principal aspect. The encoding of Minsky machines in MAELL requires such nonstandard computational steps. On the other hand, every action that is expressible in MSEL corresponds to a natural single-threaded computational feature: $\otimes$ corresponds to separation, $\forall$ to concurrency, $?$ to placement or context switching, and $!$ to waiting or sequentialization.

There are a number of other logical systems that provide some kind of staged computation or provide mechanisms for testing for properties of a subset of linear resources. Probably the most widely known is the computational monad approach pioneered by Concurrent LF (Watkins et al., 2003), which was then implemented in the Lollimon logic programming language (López et al., 2005). This idea can be seen as extending linear logic with a lax modality and then giving it an operational mode of computation up to quiescence. As far as we are aware, the decidability of the propositional fragment of this logic is still open. In the non-linear world there are far too many techniques for incorporating staged computations to survey in this article. We simply note that most propositional non-linear modal logics turn out to be decidable.

Any MSEL with only linear subexponentials will be decidable because contraction is the only way to make a proof infinitely deep. This work leaves open the questions of decidability of an arbitrary MSEL with a two-element signature where at least one subexponential is unbounded. Such logics lie between $\text{MSEL}_{\Xi}$ and MELL itself and may have easier decision problem than the latter.

Finally, this undecidability result should be taken as a word of caution for the increasingly popular uses of SEL as a logical framework for the encodings of other systems (Nigam et al., 2013; Nigam et al., 2011, e.g.). If one is to avoid encoding a decidable problem in terms of an undecidable one, subexponentials must be used very carefully.

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References


